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On Existence of Solutions to Differential Equations or Inclusions Remaining in a Prescribed Closed Subset of a Finite-Dimensional Space

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This paper is devoted to existence of trajectories to differential equations and inclusions remaining in a given closed (or compact) set. Topological properties of the boundary of this set enables us to conclude the existence of such trajectories in many cases. © 2002 Elsevier Science (USA)

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0. INTRODUCTION

In the paper we deal with first-order differential inclusions

$$\dot{x}(t) \in F(x(t)) \tag{1}$$

and, in particular, equations

$$\dot{x}(t) = f(x(t)) \tag{2}$$

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in $\mathbb{R}^n$. Throughout the paper we shall denote by $S_F(x_0)$ the set of all (absolutely continuous) solutions to the following Cauchy problem:

$$\dot{x}(t) \in F(x(t)) \quad \text{for a.e. } t \geqslant 0, \qquad x(0) = x_0 \in \mathbb{R}^n. \tag{3}$$

A closed set $K \subset \mathbb{R}^n$ being given, we study the problem of existence of trajectories of above differential inclusions (equations) remaining in K:

$$\text{P} \quad \begin{cases} \text{Does there exist } x_0 \in K \text{ and a solution } x(\cdot) \text{ to (3)} \\ \text{such that } \forall t \geqslant 0,\ x(t) \in K? \end{cases}$$

This problem addressed first by Poincaré, has many applications. For instance, we refer the reader to the monography [3] and the references therein for a large field of applications.

It is well known that problem P has a positive answer for any $x_0 \in K$, when the boundary of K is the level set of a Lyapunov function associated with the differential equation or inclusion. Necessary and sufficient conditions for positive answer to P for any $x_0 \in K$ have been obtained using the following tangential condition[2] on the boundary ∂K of K:

$$F(x) \cap T_K(x) \neq \emptyset \qquad \text{for every } x \in \partial K, \tag{4}$$

where

$$T_K(x) := \left\{ v \in \mathbb{R}^n \,\middle|\, \liminf_{h \to 0^+} \mathit{dist}(x + hv, K)/h = 0 \right\}$$

stands for the Bouligand contingent cone to K in x (cf. [3, Theorem 3.3.2]). For differential equations such results have been proved first in [24], see also [7].

For differential equations having unique solutions, Ważewski [30, Theorem 2] proved a powerful result which gives rather general conditions on behaviour of trajectories on ∂K implying an existence of a solution remaining forever in K. So let us introduce the following subsets of the boundary ∂K of the set K:

$$K_{\mathrm{s}}(F) := \{x_0 \in \partial K \mid \forall x \in S_F(x_0)\text{: } x \text{ leaves } K \text{ immediately}\},$$

$$K_{\mathrm{e}}(F) := \{x_0 \in \partial K \mid \exists x \in S_F(x_0)\text{: } x \text{ leaves } K \text{ immediately}\}.$$

[2] Sometimes this condition is expressed in the form

$$\inf_{v \in F(x)} \langle D^- d_K(x), v \rangle \leqslant 0 \text{ for every } x \in \partial K,$$

where $D^- d_K(\cdot)$ denotes the lower Dini derivative of the distance function to K.

Differential inclusions as well as differential equations without uniqueness bring us some difficulties. In particular, it may occur that there are some trajectories starting from a boundary point and leaving K immediately and some others which go inside, simultaneously. There are several papers dealing with the class of problems without uniqueness of solutions (see e.g. [5, 6, 19]) but, as a necessary assumption, the authors have considered only situations where the sets of so-called "egress" and "strictly egress" points are equal. In Section 2 we give an example (see Example 2.2) showing an importance of such restriction. The common point of these works lies in using the so-called multivalued retraction, which will be discussed in Subsection 2.2.

On the contrast, Cardaliaguet [8, 9], has obtained the result dealing with differential inclusions, where the assumption on coinciding of egress and strictly egress points has been dropped, but considerations have been restricted to narrow, from the topological point of view, connectedness condition (see Proposition 1.6). Note that the assumption on behaviour of a map on a boundary of a set, proposed by the author, can be described and verified by the use of contingent cones.

The aim of the present paper is to propose some more general sufficient conditions for solvability of problem P. We shall do it without limitation to connectedness, without assuming $K = \overline{Int\,K}$ and, in many cases, without any regularity assumptions on ∂K. We mainly propose two different—and often independent—methods for solving P:

Using homology theory: In this part, we obtain new results for continuous differential equations and we solve P when the Čech-homology groups $\check{H}(K_e(F))$ and $\check{H}(K)$ of K_e and K are not isomorphic. Extensions to differential inclusions are also obtained. For any solution x to (1) define

$$\tau_K(x) = \inf\{t > 0 \mid x(t) \notin K\}, \tag{5}$$

the first exit time to K. The main result is the following.

THEOREM A. *Let K be a closed subset of $\mathbb{R}^n$ and $F : \mathbb{R}^n \multimap \mathbb{R}^n$ a Marchaud map. Assume that $K_e(F)$ is closed and*

$$\textit{for every } x_0 \in K_e(F) \textit{ and every } x \in S_F(x_0),\ x([0, \tau_K(x)]) \subset K_e(F). \tag{6}$$

If $\check{H}(K_e(F))$ and $\check{H}(K)$ are not isomorphic, then problem P *has a solution.*

We give an example showing that there are natural situations covered by this new approach in contrast to the "multivalued retraction" idea proposed in previous papers (see e.g. [5, 19]).

Using retracts: When $K_s(F)$ is not a strong deformation retract of K, problem P has a positive answer for differential inclusions. This enables us to give several results; among them there is the following

THEOREM B. *Let K be a closed subset of $\mathbb{R}^n$ and F a Marchaud locally Lipschitz map. Assume that $K_s(F)$ is closed, $M := K_e \backslash K_s(F)$ is a compact proximal retract and there exists a Lipschitz selection $h: M \to \mathbb{R}^n$ of $F(\cdot) \cap T_K(\cdot)$.*

If $K_s(F)$ is not a strong deformation retract of K, then problem P *has a solution for F.*

This method is mainly based on the existence of a single-valued Lipschitz selection of the right-hand side $F$of (1). As a byproduct, we obtain some Lipschitz selection results. We study an important class of problems, where we drop the narrowing assumption that no boundary point is of egress and ingress simultaneously.

All the results obtained in the paper are illustrated by some examples.

Let us explain how the paper is organized. Section 1 contains some notations, definitions and preliminaries together with the Ważewski retract principle. Section 2 is devoted to a sufficient condition for solving P with the use of homology groups. In Section 3, various sufficient conditions are established through strong deformation retracts. Section 4 is intended to investigate properties of the set of initial conditions solving P. Appendix A provides a short exposition of results and notions of algebraic topology used in Section 2. Appendix B contains the proof of some technical lemma.

1. PRELIMINARIES

Throughout the paper, by $Int\,A$, $cl\,A$ (or $\bar{A}$) and ∂A we denote, respectively, the interior, the closure and the boundary of a subset A of a metric space X. An open ball centred in x_0 and with a radius r is denoted by $B(x_0, r)$. We also use a notation $|\cdot|$ for a euclidean norm. By $d_M(x)$ (or $dist(x, M)$) we denote a distance from a point x to a closed set M. The distance between two sets N, M will be always denoted by $dist(N, M) := \inf\{d_M(x) \mid x \in N\}$.

All spaces are assumed to be metric and single-valued maps to be continuous. By a multivalued map (denoted with the symbol $\multimap$) $\varphi: X \multimap Y$ we will always mean a map with nonempty closed values for all $x \in X$.

1.1. Solutions to Differential Inclusions

Let us start with some notations and observations which will be needed in the following sections.

DEFINITION 1.1. The set-valued map $F : \mathbb{R}^n \multimap \mathbb{R}^n$ is a *Marchaud map* if and only if F is upper semicontinuous (in short: u.s.c.) with compact convex values and linear growth (that is, there is a constant $c > 0$ such that $|F(x)| := \sup\{|y| \mid y \in F(x)\} \leqslant c(1 + |x|)$, for every x).

It is known [3, Theorem 3.3.5] that for each $x_0 \in \mathbb{R}^n$ there is an absolutely continuous solution to the Cauchy problem (3). Moreover, it satisfies the estimates

$$\text{for all } t \geqslant 0 \qquad |x(t)| \leqslant (|x_0| + 1)e^{ct} \tag{7}$$

and

$$\text{for a.e. } t \geqslant 0 \qquad |\dot{x}(t)| \leqslant c(|x_0| + 1)e^{ct}. \tag{8}$$

Take $b > c$. The set of all absolutely continuous solutions to (3) is viewed as a subspace of the Banach space

$$C := \left\{ x \in C([0, \infty), \mathbb{R}^n) \,\middle|\, \sup_{t \geqslant 0} |x(t)|e^{-bt} < \infty \right\}$$

equipped with the norm

$$\|x\|_C := \sup_{t \geqslant 0} |x(t)|e^{-bt}.$$

LEMMA 1.2 (Andres *et al.* [1]; Aubin [3, Theorem 2.4.4, Corollary 5.3.3]). *If F is a Marchaud map, then $S_F : \mathbb{R}^n \multimap C$ is u.s.c. with nonempty compact R_δ values.*[3] *Moreover, if F is Lipschitz, then S_F is also lower semicontinuous.*

Denote $S_F(K) := \bigcup_{x \in K} S_F(x)$. Recall that the (exit) function $\tau_K : S_F(K) \to [0, \infty]$, defined by (5) is u.s.c. (see [3, Lemma 4.2.2]). So using the function τ_K we can deduce the following result concerning the differential inclusion:

$$\dot{x}(t) \in -F(x(t)). \tag{9}$$

[3] A space X is a compact R_δ-set provided it is homeomorphic to an intersection of a decreasing sequence of compact contractible spaces. In particular, it is acyclic.

LEMMA 1.3 (Cardaliaguet [8, 9, Proposition 3.1]). *If K is compact and F is a Marchaud map, problem* P *for* (1) *has a solution if and only if it has a solution for* (9).

In fact, there are points in K being simultaneously initial points of trajectories solving P for (1) and trajectories solving P for (9).

1.2. Subsets of ∂K and the Ważewski topological principle

In 1947 Ważewski proved the following result solving P, when condition (4) is not satisfied on the whole boundary of a set.

THEOREM 1.4 (Ważewski Topological Principle [30, Theorem 2]). *Let $K = \overline{\mathrm{Int}\, K} \subset \mathbb{R}^n$, Ω be an open neighbourhood of K in $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^n$ be a map such that the Cauchy problem for* (2) *has a unique solution defined on the whole* $[0, \infty)$, *for every initial point $x_0 \in \Omega$. Assume that each trajectory starting from* Int K *and attaining ∂K leaves K immediately. Denote*

$$S := \{x_0 \in \partial K \mid \textit{there is } t_0 > 0 \textit{ and a solution } x \textit{ to } (2) \textit{ such that}$$
$$x([0, t_0)) \subset \mathrm{Int}\, K \textit{ and } x(t_0) = x_0\}.$$

If $Z \subset \mathrm{Int}\, K \cup S$ is such that $Z \cap S$ is a retract of S and not of Z, then there is a trajectory starting from $Z \cap \mathrm{Int}\, K$ and staying in Int K, *hence, the problem* P *has a positive answer.*

In particular, for $Z = \mathrm{Int}\, K \cup S$, there is a solution staying in Int K *whenever S is not a retract of Z.*

In the present paper we deal with more general differential problems, where a behaviour of a map on ∂K is more difficult to be described. Moreover, since we do not restrict our considerations to open sets, we use, instead of the set S, some other subsets of ∂K: the subsets $K_s(F)$ and $K_e(F)$ (shortly denoted K_s and K_e when there is no ambiguity) defined in the introduction of the present paper.

We introduce the following notations for subsets of the boundary of K (cf. [9, 26]):

$$K_{\Rightarrow} := \{x \in \partial K \mid F(x) \cap T_K(x) = \emptyset\},$$
$$K_{\Leftrightarrow} := \{x \in \partial K \mid F(x) \cap T_{\partial K}(x) \neq \emptyset\},$$
$$K_{\Leftarrow} := \{x \in \partial K \mid F(x) \cap T_{\mathbb{R}^n \setminus K}(x) = \emptyset\}.$$

One can check (see [8, Lemma 3.1]) that

$$K_{\Rightarrow} \subset K_s \subset \overline{K_{\Rightarrow}}.$$

Note that for F-Lipschitz, using the Invariance Theorem (see e.g. [3, Theorem 5.3.4]), one can describe K_e by suitable contingent cones conditions. Indeed, in this case, if K_e is closed,

$$\partial K \setminus K_e := \{x \in \partial K \mid \exists U_x \subset \partial K, x \in U_x \forall z \in U_x : F(z) \subset T_K(z)\}.$$

In [8] there is the following contingent cone characterization of the set K_s:

PROPOSITION 1.5 (cf. Cardaliaguet [8, Proposition 3.1]). *Let K be closed and F a Marchaud map locally Lipschitz around $x \in \overline{K_\Rightarrow} \setminus K_\Rightarrow$.*

If

$$F(x) \cap ((\mathbb{R}^n \setminus T_{\mathbb{R}^n \setminus K}(x)) \cup T_{\partial K \setminus K_\Rightarrow}(x)) = \emptyset,$$

then $x \in K_s$.

If

$$F(x) \cap (\mathbb{R}^n \setminus T_{\mathbb{R}^n \setminus K}(x)) \neq \emptyset \text{ or } F(x) \cap T_{K_\Rightarrow}(x) = \emptyset,$$

then $x \notin K_s$.

Let us add that for single-valued Lipschitz right-hand sides F, $K_s = K_e$. Using some properties of the set K_s, Cardaliaguet has proved the following.

PROPOSITION 1.6 (Cardaliaguet [8, 9, Theorem 2.1]). *Let K be a closed convex subset of $\mathbb{R}^n$ and F a Marchaud map. If the set K_s is closed and not connected, then problem* P *has a positive answer.*

Remark 1.7. Note that the assumption on convexity of K may be weakened in the following way (see [8, Corollary 2.1]).

For any $x \in \partial K$ such that $F(x) \cap T_{\partial K}(x) \neq \emptyset$,

there exist a closed neighbourhood U of x in $\mathbb{R}^n$ and a diffeomorphism $\phi : U \to \mathbb{R}^n$ on its image such that $\phi(U \cap \partial K)$ is convex. (10)

In the proof of Proposition 1.6 the author has used upper semicontinuity of the map τ_K. In the sequel we shall use some further properties of this map.

LEMMA 1.8. *Let K be a closed subset of $\mathbb{R}^n$, F a Marchaud map and $K_e = K_s$ be closed. Assume that all trajectories starting from K leave it. Then the function $\tau_K : S_F(K) \to [0, \infty)$ defined by* (5) *is continuous.*

Proof. From Lemma 4.2.2 in [3], it follows that τ_K is u.s.c. So, let us show that it is l.s.c., namely:

For every $x \in S_F(K)$ and every $\varepsilon > 0$, there is $\delta > 0$ such that $\tau_K(z) > \tau_K(x) - \varepsilon$, for each $z \in S_F(K)$ with $\|z - x\|_C < \delta$.

Let $x \in S_F(K)$ be an arbitrary solution. If $\tau_K(x) = 0$, then $\tau_K(z) > \tau_K(x) - \varepsilon$, for any $z \in S_F(K)$. So it is sufficient to assume that $\tau_K(x) > 0$ and $\varepsilon > 0$ is such that $\tau_K(x) - \varepsilon > 0$.

Suppose, on the contrary, that, for any $n \geqslant 1$, there is $z_n \in S_F(K)$ with $\|z_n - x\| < \frac{1}{n}$ on $[0, \tau_K(x)]$ and $\tau_K(z_n) \leqslant \tau_K(x) - \varepsilon$.

Since $[0, \tau_K(x)]$ is compact, we can assume that $\tau_K(z_n) \to t_0 \in [0, \tau_K(x) - \varepsilon]$. Notice that

$$|x(t_0) - z_n(\tau_K(z_n))| \leqslant |x(t_0) - x(\tau_K(z_n))| + |x(\tau_K(z_n)) - z_n(\tau_K(z_n))|,$$

which implies that $x(t_0) \in K_e = K_s$, since $z_n(\tau_K(z_n)) \in K_e$ and K_e is closed. So $t_0 = \tau_K(x)$; a contradiction. ∎

Define the function $\rho_K : S_F(K) \to [0, \infty)$,

$$\rho_K(x) := \inf\{t > 0 \mid x(t) \in K_e\}.$$

LEMMA 1.9. *Assume that $Z \subset K$ and no trajectory starting from Z remains in K. The function $\rho_K(\cdot)$ is l.s.c. on Z provided*

$$\textit{for each } x_0 \in \overline{K_e} \backslash K_e \textit{ and } x \in S_F(x_0),\ x([0, \infty)) \cap K_e = \emptyset. \tag{11}$$

Proof. Take $x \in S_F(Z)$, $\varepsilon > 0$ and assume, on the contrary, that there is a sequence $z_k \to x$ with $\rho_K(z_k) \leqslant \rho_K(x) - \varepsilon$. Analogously as in the proof of Lemma 1.8, we obtain that $x(t_0) \in cl\{z(\rho_K(z)) \mid z \in S_F(Z)\}$, for some $t_0 \leqslant \rho_K(x) - \varepsilon$. Therefore $x(t_0) \in \overline{K_e} \backslash K_e$ and, by (11), we have a contradiction. ∎

Remark 1.10. Condition (11) is obviously satisfied if $Z \supset K_e$ and K_e is closed. There are important examples, where (11) also holds for nonclosed K_e. For instance, in considerations in [2], $\overline{K_e} \backslash K_e$ is a singleton in which F is equal to zero. Second example can be found in [22, Sect. 5]. Notice also that if $\overline{K_e} \backslash K_e \neq \emptyset$, then assumption (11) implies a solvability of problem P. Therefore, (11) will be used only in results localizating initial points of trajectories solving P (see Subsection 2.2); in other results we will assume that the set K_e is closed.

2. ADMISSIBLE MAPS METHOD

2.1. Homology Approach

At first we prove Theorem A formulated in the Introduction, the main result of this section. For the convenience of the reader, we include the relevant material from algebraic topology in Appendix A postponed at the end of the paper.

Proof of Theorem A. Assume, on the contrary, that problem P has no solution. Consider the multivalued homotopy $H : K \times [0,1] \multimap K$,

$$H(x_0, t) := \bigcup_{x \in S_F(x_0)} x([t\rho_K(x), t\tau_K(x)]).$$

It can be described as the following composition:

$$K \times [0,1] \overset{S_F \times id}{\multimap} S_F(K) \times [0,1] \overset{J \times id}{\multimap} S_F(K) \times [0,\infty) \times [0,1] \overset{k}{\to} K,$$

where $(S_F \times id)(x_0, t) := S_F(x_0) \times \{t\}$, $(J \times id)(x, t) := \{x\} \times [\rho_K(x), \tau_K(x)] \times \{t\}$ and $k(x, s, t) := x(st)$.

We check that J is an admissible map (see Appendix A). Indeed, since ρ_K is l.s.c. and τ_K u.s.c., J is u.s.c. in meaning of multivalued maps. Obviously, values of J are compact and convex, so J is admissible, by Remark 6.2.

Note that from (6) it follows that $x([\rho_K(x), \tau_K(x)]) \subset K_e$, for every $x \in S_F(K)$.

From Lemma 1.2 and Remark 6.2 it follows that S_F is admissible. Hence, the map H, as a composition of admissible maps, is admissible. Moreover, it joins $H(\cdot, 0) = id_K$ with $H(\cdot, 1) = i \circ \Phi : K \multimap K$, where $\Phi : K \multimap K_e$,

$$\Phi(x_0) := \bigcup_{x \in S_F(x_0)} x([\rho_K(x), \tau_K(x)]), \tag{12}$$

which implies that $H(x_0, t) \ni x_0$, for every $x_0 \in K_e$ and $t \in [0,1]$.

Consider the diagram

$$\begin{tikzcd}
K_e \arrow[r, "i"] \arrow[d, multimap, "\Phi \circ i"'] & K \arrow[dl, multimap, "\Phi"] \arrow[d, multimap, "i \circ \Phi"] \\
K_e \arrow[r, "i"] & K
\end{tikzcd}$$

From Proposition 6.5 used for H, $Id_{\check{H}(K)} \in (i \circ \Phi)_*$, which means that $Id_{\check{H}(K)} = i_* q_* p_*^{-1}$ for some selected pair (p, q) of Φ (cf. (A.1)) and hence, i_* is onto. On the other hand, since $id_{K_e} \subset \Phi \circ i$, from (A.2) one obtains $Id_{\check{H}(K_e)} = \bar{q}_* \bar{p}_*^{-1} i_*$ for some selected pair $(\bar{p}, \bar{q}) \subset \Phi$. This implies that i_* is injective and so, an isomorphism; a contradiction. ∎

Remark 2.1. As one can see in the proof, we have shown a little more than in the statement. Namely, we have proved that, under the assumptions of Theorem A, if the inclusion $i : K_e \to K$ does not induce an isomorphism $(i_* : \check{H}(K_e) \to \check{H}(K))$, then problem P has a solution.

The following example shows that without assumption (6) our sufficient condition proposed in Theorem A may be false. It explains a difficulty which arises when a boundary point is of egress and ingress simultaneously.

EXAMPLE 2.2. Let

$$K := ([-1, 2] \times [-3, 3]) \setminus \{(x, y) \in \mathbb{R}^2 \mid x > 0, -x^2 < y < x^2\}.$$

Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$,

$$f(x, y) := \begin{cases} (1, 2\sqrt{y}), & 0 \leqslant y \leqslant 1, \\ (1, -2\sqrt{-y}), & -1 \leqslant y < 0, \\ (1, 2\sqrt{2 - y}), & 1 < y \leqslant 2, \\ (1, -2\sqrt{y - 2}), & y > 2, \\ (1, -2\sqrt{y + 2}), & -2 \leqslant y < -1, \\ (1, 2\sqrt{-y - 2}), & y < -2. \end{cases}$$

Notice that K is contractible (in particular, connected), while K_s and $K_e = K_s \cup \{(0, 0)\}$ are closed disconnected, and hence both $\check{H}(K_s)$ and $\check{H}(K_e)$ are not isomorphic to $\check{H}(K)$. Moreover, f is continuous, but problem P has no solution.

As one can surmise, the reason is connected with properties of f in (0,0). Indeed, there are solutions starting from (0,0) and going into K, while $(x(t), y(t)) = (t, 0)$ leaves K immediately. Notice that (0,0) is the only boundary point (for $x \geqslant 0$) without uniqueness of trajectories starting from it. Moreover, it is also the only point where (10) is not satisfied, so we have given an example showing necessity of (10) in Proposition 1.6.

2.2. *On Multivalued Retracts Approach*

Some papers dealing with problems without uniqueness of solutions have used the notion of multivalued retraction. Below we show that such results are consequences of Theorem A. Indeed, Theorem A easily leads us to the following.

COROLLARY 2.3. *Let K be a closed subset of $\mathbb{R}^n$ and $F:\mathbb{R}^n \multimap \mathbb{R}^n$ a Marchaud map. Assume that K_e is closed and* (6) *is satisfied. Then, if there is no admissible multivalued retraction*[4] *of K onto K_e, the problem* P *has a solution. In particular, if K is connected and K_e is disconnected, then problem* P *is solved.*

Proof. Assume, on the contrary, that no trajectory remains forever in K. Define, as above, the map $\Phi : K \multimap K_e$ (see (12)). Since Φ is an admissible upper semicontinuous map onto K_e, it is a suitable multivalued retraction. Using the lemma below, we obtain the second part of the statement. ∎

LEMMA 2.4. *If $\varphi : X \multimap Y$ is semicontinuous with connected values, then $\varphi(A)$ is connected, for each connected subset A of X.*

The technique described above allows us to obtain also the result on localization of initial points of trajectories solving P, which generalizes results of e.g. [5, 6, 19], etc.

COROLLARY 2.5. *Let K be a closed subset of $\mathbb{R}^n$ and $Z \subset K$ an arbitrary subset. Assume that $F:\mathbb{R}^n \multimap \mathbb{R}^n$ is a Marchaud map satisfying* (11) *and*

$$\text{for each } x_0 \in Z \text{ and } x \in S_F(x_0),\ x([\rho_K(x), \tau_K(x)]) \subset K_e. \tag{13}$$

If there is an admissible multivalued retraction of K_e onto $Z \cap K_e$ and there is no admissible multivalued retraction of Z onto $Z \cap K_e$, then there is a trajectory $x(\cdot)$ starting from $Z \backslash K_e$ and solving problem P.

Proof. Assuming that, for every $x_0 \in Z \backslash K_e$, each trajectory starting from x_0 leaves K, we construct, as above, an admissible map $\Phi : Z \multimap K_e$, given by (12) (we use Lemma 1.9 in proving the upper semicontinuity of Φ, since K_e may be not closed). Composing it with the admissible multivalued retraction $H : K_e \multimap K_e \cap Z$, we obtain $H \circ \Phi : Z \multimap K_e \cap Z$, an admissible multivalued retraction, contradicting our assumption. ∎

[4] We say that $\Phi : X \multimap M \subset X$ is a *multivalued retraction*, if Φ is u.s.c. with compact values and $x \in \Phi(x)$, for every $x \in M$.

Remark 2.6. Note that the notion of multivalued retraction without any regularity assumptions (as in e.g. [5, 6, 19] is too general and it seems to be not appropriate. It is well known that one can find a multivalued retraction with connected values even of the ball onto its boundary. Admissibility of a map is just a suitable property, which is useful in topological fixed point theory and some related topics (see [16] and references therein). Summing up, we have essentially improved previously known results and we have found more adequate sufficient condition for solving problem P, also in a multivalued case.

Assumption (13) in Corollary 2.5 is slightly weaker than (6) and allows us to admit an arbitrary behaviour of F on parts of K_{e} which are out of interest.

EXAMPLE 2.7. Consider the following system in $\mathbb{R}^n$:

$$\dot{x}_i = x_i f_i(x), \qquad i = 1, \ldots, n \tag{14}$$

with assumptions:

(A1) The map $f = (f_1, \ldots, f_n) : \mathbb{R}^n_+ \to \mathbb{R}^n$ is continuous, where $\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geqslant 0$ for every $i = 1, \ldots, n\}$.

(A2) There is $c > 0$ and, for every x_i, there is a level $0 < d_i < c$ such that

$$\begin{cases} d_i f_i(x) > 0, & |x| < c,\ x \in K, \\ d_i f_i(x) = 0, & |x| = c,\ x \in K, \\ d_i f_i(x) < 0, & |x| > c,\ x \in K, \end{cases}$$

where $|x| := |x_1| + \cdots + |x_n|$ and $K := \{x \in \mathbb{R}^n_+ \mid x_i \geqslant d_i$ for every $i = 1, \ldots, n\}$.

(A3) For $x \in X_i := \{x \in K \mid |x| = c,\ x_i = d_i\}$, we have $\sum_{j=1}^n x_j f_j(x) \geqslant 0$.

(A4) f is Lipschitz on a neighbourhood of $\bigcup_{i=1}^n X_i$.

Under assumptions (A1)–(A4), there is a trajectory starting from the set

$$Z := \bigcup_{i=1}^{n} \{x \in K \mid |x| \leqslant c \text{ and } x_i = d_i\}$$

and remaining in K. To prove the statement, notice that

$$\partial K = \bigcup_{i=1}^{n} \{x \in K \mid x_i = d_i\},$$

and easily verify that

$$K_{\Rightarrow} = \{x \in \partial K \mid |x| > c\} \subset K_s \subset \{x \in \partial K \mid |x| \geqslant c\}.$$

We check that $\{x \in \partial K \mid |x| = c\} \subset K_s$.

Let $x = (x_1, \ldots, x_n) \in K$ be a point with $x_1 + \cdots + x_n = c$ and $x_i = d_i$. By (A3), $\langle (1, \ldots, 1),\ F(x) \rangle \geqslant 0$, where $F(x) \coloneqq (x_1 f_1(x), \ldots, x_n f_n(x))$. Notice that $(1, \ldots, 1)$ is a normal vector to the hyperplane $x_1 + \cdots + x_n = c$. Note also that $F_i(x) = x_i f_i(x) = 0$ (assumption (A2)). Therefore, one can check that $F(x) \notin \mathbb{R}^n \setminus T_{\mathbb{R}^n \setminus K}(x)$ and $F(x) \notin T_{\partial K \setminus K_{\Rightarrow}}(x)$.

Since f is Lipschitz near x, Proposition 1.5 yields that $x \in K_s$.

Finally, notice that $Z \cap K_s$ is a retract of K_s but there is no admissible multivalued retraction of Z onto $Z \cap K_s$. Now, by Corollary 2.5, the proof is complete.

2.3. *On Property* P *for F and $-F$*

One of the extensions of the Ważewski retract method (Theorem 1.4) has been done for dynamical systems and started the method which is now called the Conley index theory (see [10]). It has been developed since 1970s in many directions. There are some papers also dealing with multivalued flows ([20, 23], see also [21] and references therein) which can be generated by continuous differential equations or differential inclusions.

To make our consideration more complete, we compare briefly the results presented in the paper with Conley index approach. Below we omit definitions of basic notions of the Conley theory (the reader is referred to e.g. [21] for details).

In a compact set K, according to Lemma 1.3, the problem P and the problem of finding a trajectory remaining in K for every $t \in \mathbb{R}$ are equivalent.

Denote by I the maximal subset of K such that, for every $x_0 \in I$, there is a trajectory x with $x((-\infty, \infty)) \subset K$ and passing through x_0 namely, in x_0 problem P is simultaneously solved for F and $-F$.

Proposition 2.8 (Kunze [20, Theorem 5.3.1]). *If K is a compact isolating neighbourhood of I and a generalized Conley homotopy index $H(I)$ is different from the trivial homotopy type $\bar{0}$, then $I \neq \emptyset$. In particular, problem* P *has a solution.*

The present paper deals with sets which may be noncompact and not necessarily with nonempty interior, in the contrast to isolated neighbourhoods. Moreover, we do not assume that the largest invariant subset I of K does not intersect ∂K. Although in the fixed point and the Conley index theories such an assumption is usually used and natural, it may bring some difficulties in verification in concrete differential problems, especially in a multivalued case. In fact, one should examine on $\partial K \setminus K_e$ behaviour of whole trajectories for F and $-F$.

3. APPROACH THROUGH RETRACTS

In this section, we give some sufficient conditions for solving P using notions of retracts[5] and strong deformation retracts[6] (for a topological background see e.g. [13]). To construct suitable retractions or homotopies we use the Lipschitz regularity of F allowing us to find suitable Lipschitz selections. To guarantee some of their necessary properties, some additional assumptions on F are needed. Nevertheless, these assumptions are satisfied in single-valued case automatically, so our results are generalizations of Ważewski's theorem (Theorem 1.4).

3.1. On Solvability of P *for Closed Sets*

Now we prove our first sufficient condition for solving P in a Lipschitz case.

PROPOSITION 3.1. *Let K be a closed subset of $\mathbb{R}^n$ and F a locally Lipschitz Marchaud map such that $K_e = K_s$ is closed.*
If K_e is not a strong deformation retract of K, then problem P *has a solution.*

Proof. Assume, by contradiction, that problem P has no solution and take a locally Lipschitz selection f of F. One has that there is no solution to problem P for f and $K_e(f) = K_e(F) = K_s$. Define a homotopy $H : K \times [0,1] \to K$,

$$H(x_0, \lambda) := x^0(\lambda \tau_K(x^0)),$$

where x^0 is the unique solution for f starting from x_0.

Since f is locally Lipschitz (we have continuous dependence on initial conditions), assumptions on K_s and K_e (see Lemma 1.8) imply that H is continuous.

Notice that

(i) $H(x_0, 0) = x_0$, for every $x_0 \in K$,

(ii) $H(x_0, 1) \in K_e$, for every $x_0 \in K$,

(iii) $H(x_0, \lambda) = x_0$, for every $x_0 \in K_e$ and $\lambda \in [0,1]$.

[5] A closed subset M of a space X is said to be a *retract* of X provided there exists a map $r : X \to M$ such that $r(x) = x$, for every $x \in M$.

[6] $M \subset X$ is a *strong deformation retract* of X if there is a homotopy $h : X \times [0,1] \to X$ such that $h(x,0) = x$, $h(x,1) \in M$, for all $x \in X$, and $h(x,t) = x$, for each $x \in M$ and $t \in [0,1]$.

Hence, H is a strong deformation of K onto K_e which contradicts our assumption. The proof is complete. ∎

As we can see in the following example, the closedness assumption on K_e is necessary.

EXAMPLE 3.2. Let $\mathbb{R}^2 \supset K = \overline{B((0,0),1)} \setminus \{(x,y) \in \mathbb{R}^2 \mid x > 0 \text{ and } -x < y < 0\}$ and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a constant map $f(x,y) = (0,1)$. Then f is Lipschitz,

$$K_e = K_s = \{(x,y) \in \mathbb{R}^2 | -1 \leqslant x \leqslant 1 \text{ and } y = \sqrt{1-x^2}\}$$
$$\cup \{(x,y) \in \mathbb{R}^2 \mid 0 < x \leqslant \tfrac{\sqrt{2}}{2} \text{ and } y = -x\}.$$

Moreover, K_e is disconnected (so, it is not a retract of K) and there are no solutions remaining forever in K. Notice that $(0,0) \in \overline{K_e} \setminus K_e$.

From Proposition 3.1, one can immediately obtain

COROLLARY 3.3. *If $K = \overline{\mathrm{Int}\, K}$ and $K_s = \partial K$, then* P *has a solution for F.*

Indeed, ∂K is not a retract of $K = \overline{\mathrm{Int}\, K}$ (see [13, p. 341]).

Let us add the following simple illustration of Proposition 3.1.

COROLLARY 3.4. *Let $\Omega \subset \mathbb{R}^{n-1}$ be an open bounded subset and $K = \mathbb{R} \times \bar{\Omega}$. Assume that $F : \mathbb{R}^n \multimap \mathbb{R}^n$ is a locally Lipschitz Marchaud map such that $K_e = K_s$ and $\{a\} \times \partial\Omega \subset K_e$ for some $a \in \mathbb{R}$. Then problem* P *has a solution in K.*

Proof. Indeed, supposing that all trajectories leave K, we proceed as in the proof of Proposition 3.1 and find a retraction r of K onto K_e. Consider the map $r_1 : \bar{\Omega} \to K_e$, $r_1(y) := r(a,y)$ and the composition $r_2 := \pi \circ r_1$, where $\pi(x,y) := y$, for every $(x,y) \in K$. Then r_2 is a retraction of $\bar{\Omega}$ onto $\partial\Omega$; a contradiction. ∎

EXAMPLE 3.5. Let K be as above and let F satisfy the condition $K_e = K_s = \partial K$. Then problem P is solved in K.

Using the Lipschitz selection technique and similar arguments as in the proof of Proposition 3.1 we can easily obtain the following multivalued generalization of Ważewski's theorem.

PROPOSITION 3.6. *Let $K = \overline{\mathrm{Int}\, K}$ and F be a Marchaud locally Lipschitz map such that each trajectory starting from $\mathrm{Int}\, K$ and reaching ∂K leaves K*

immediately. Let S be as in Theorem 1.4 *and $Z \subset Int\, K \cup S$ be an arbitrary subset.*

If $Z \cap S$ is a strong deformation retract (*resp. retract*) *of S and it is not a strong deformation retract* (*resp. retract*) *of Z, then problem* P *has a solution. More precisely, there is a trajectory starting from $Z \backslash S$ and remaining in Int K.*

Proof. Assuming that P has no solution starting from $Z \backslash S$ and taking a locally Lipschitz selection f of F, we can define a homotopy (as in the proof of Proposition 3.1) $H: Z \times [0,1] \to Z$, $H(x_0, \lambda) := x^0(\lambda \tau_K(x^0))$. It deformates Z onto $H(Z \times \{1\})$. Since we do not assume that K_e is closed, we should check that $\tau_K(\cdot)$ is continuous on $S_f(Z)$.

Notice that $\tau_K(x) = \rho_K(x)$ for any $x \in S_f(Z)$. So, it is sufficient to check that $\rho_K(\cdot)$ is l.s.c. on $S_f(Z)$.

For every trajectory x starting from $Z \cap S$ we have $\rho_K(x) = 0$ and hence, $\rho_K(\cdot)$ is l.s.c. in x. Let $x_0 \in Int\, K \cap Z$ and x^0 be a trajectory starting from x_0 with $\rho_K(x^0) > 0$. As in the proof of Lemma 1.9, assuming that $\rho_K(\cdot)$ is not l.s.c. in x^0, we can prove that $x^0(t_0) \in S$ for some $t_0 < \rho_K(x^0)$; a contradiction.

Let $h: S \times [0,1] \to S$ be a homotopy with $h(\cdot, 0) = id$ deforming S onto $Z \cap S$. Defining $k: Z \times [0,1] \to Z$,

$$k(x_0, t) := \begin{cases} H(x_0, 2t), & 0 \leqslant t \leqslant 1/2, \\ h(H(x_0, 1), 2t-1), & 1/2 \leqslant t \leqslant 1, \end{cases}$$

we obtain a homotopy deforming Z onto $Z \cap S$; a contradiction.

The proof for retractions are analogous. Instead of the homotopy k, we construct a retraction $r \circ H(\cdot, 1): Z \to Z \cap S$, where $r: S \to Z \cap S$ is a retraction given in assumptions. ∎

We shall end this subsection with two concrete examples.

EXAMPLE 3.7. Consider the nonlinear system

$$\begin{cases} \dot{x} \in x + F_1(x,y,z), \\ \dot{y} \in y + F_2(x,y,z), \\ \dot{z} \in -z + F_3(x,y,z), \end{cases} \tag{15}$$

where F_1, F_2, F_3 are Lipschitz maps vanishing on the sphere S^2. Let $K = cl\, B(0,1) \subset \mathbb{R}^3$, and define $F(x,y,z) := (x + F_1(x,y,z)) \times (y + F_2(x,y,z)) \times (-z + F_3(x,y,z))$.

Note that

$$
\begin{aligned}
K_{\Rightarrow} &= \{(x,y,z) \in S^2 \mid \langle F(x,y,z),(x,y,z)\rangle > 0\} \\
&= \left\{(x,y,z) \in S^2 \mid -\frac{\sqrt{2}}{2} < z < \frac{\sqrt{2}}{2}\right\}
\end{aligned}
$$

and

$$
K_{\Leftrightarrow} = \left\{(x,y,z) \in S^2 \mid z = -\frac{\sqrt{2}}{2} \text{ or } z = \frac{\sqrt{2}}{2}\right\} \subset \overline{K_{\mathrm{s}}}.
$$

Of course $K_{\mathrm{e}} = K_{\mathrm{s}} \subset K_{\Rightarrow} \cup K_{\Leftrightarrow}$. To check that K_{s} is closed, we use the Cardaliaguet's characterization (see Proposition 1.5) which says that it is sufficient to show that, for every $(x,y,z) \in \overline{K_{\Rightarrow}} \setminus K_{\Rightarrow}$,

$$
F(x,y,z) \notin (\mathbb{R}^3 \setminus T_{\mathbb{R}^3 \setminus K}) \cup T_{\partial K \setminus K_{\Rightarrow}}(x,y,z).
$$

Since obviously $F(x,y,z) \notin \mathbb{R}^3 \setminus T_{\mathbb{R}^3 \setminus K}(x,y,z)$, we check that $F(x,y,z) \notin T_{\partial K \setminus K_{\Rightarrow}}(x,y,z)$. Indeed, if it is not the case,

$$
\langle (x,y,z),(x,y,-z)\rangle = 0 \text{ and } \langle (x,y,0),(x,y,-z)\rangle \leqslant 0
$$

which is impossible. It is easy to see that K_{s} is not a retract of K. Now from Proposition 3.1, it follows that problem P has a solution.

EXAMPLE 3.8. Consider the following problem:

$$
\begin{cases}
\dot{r} \in -(r-4)\cos\varphi + F_1(r,\varphi,\psi), \\
\dot{\varphi} \in \sin\varphi + F_2(r,\varphi,\psi), \\
\dot{\psi} \in -\sin 3\psi \cos\varphi + F_3(r,\varphi,\psi)
\end{cases}
$$

on the torus $K \subset \mathbb{R}^3$ centred in the origin and giving on the plane $z = 0$ two balls $(x-4)^2 + y^2 \leqslant 1$ and $(x+4)^2 + y^2 \leqslant 1$. Let us describe it in spherical coordinates, where $\varphi \in [0, 2\pi]$ and $\psi \in [-\arcsin\frac{1}{4}, \arcsin\frac{1}{4}]$. Here F_1, F_2, F_3 are Lipschitz maps vanishing on ∂K.

One can check that K_{s} is a "half" of ∂K ($\varphi \in [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi]$) which implies that K_{s} is not a retract of K. Hence, for

$$
\begin{aligned}
F(r,\varphi,\psi) := (-(r-4)\cos\varphi + F_1(r,\varphi,\psi)) \times (\sin\varphi + F_2(r,\varphi,\psi)) \\
\times (-\sin 3\psi \cos\varphi + F_3(r,\varphi,\psi)),
\end{aligned}
$$

problem P has a solution.

3.2. On solvability of P *for sets with some smoothness of the boundary*

The subsection below concerns extension of previous results without the assumption $K_s = K_e$. We shall see that it requires more regularity of ∂K (or of some subset of ∂K).

The difficulty in finding an analogue to Proposition 3.1 consists in the fact that for a locally Lipschitz selection f of F it may occur that $K_s(f) \neq K_s(F)$. Therefore now, we would like to find sufficient conditions guaranteeing the existence of an appropriate selection of F with $K_s(f) = K_s(F)$.

Let us recall that a subset M of $\mathbb{R}^n$ is said to be a *proximal retract* (see e.g. [25]), if there is an open neighbourhood V of M such that

$$\pi_M(x) := \left\{ y \in M \mid |y - x| = \inf_{u \in M} |u - x| \right\} \text{ is a singleton} \tag{16}$$

for every $x \in V$. It means that π_M is a retraction from V onto M. One can prove that for every proximal retract M the map $T_M(\cdot)$ is lower semicontinuous on M (see e.g. [17]).

Before proving Theorem B, the main result of this section, we need the following two lemmas.

LEMMA 3.9. *If $M \subset \mathbb{R}^n$ is a compact proximal retract, then the projection π_M is Lipschitz on some open neighbourhood of M.*

This well-known Lemma is, for instance, a consequence of Theorem 1.3 in [28].

Remark 3.10. From Federer's result (see [14]) and the above lemma it follows that each $C^{1,1}$ manifold is a proximal retract.

LEMMA 3.11. *Assume that $M \subset \mathbb{R}^n$ is a compact proximal retract and F is a Lipschitz map on some open neighbourhood Ω in M. Then, for each Lipschitz selection $h : M \to \mathbb{R}^n$ of F, there is an extension $k : V \to \mathbb{R}^n$ on some open neighbourhood V of M, which is a Lipschitz selection of F.*

Proof. Let $V \supset M$ be such that (16) holds on V. Consider the map $G : V \multimap \mathbb{R}^n$,

$$G(x) := F(x) \cap dB(h(\pi_M(x)), 2d_{F(x)}(h(\pi_M(x))). \tag{17}$$

By Lemma 9.4.2 in [4] and Lemma 3.9, G is Lipschitz. Moreover, $G(x) = h(x)$ for every $x \in M$.

Now, a locally Lipschitz selection k of G exists by Aubin [4, Theorem 9.4.3], and it is an extension of h on V. ∎

Remark 3.12. Notice that in the above lemma, we can assume that M is any Lipschitz retract of some open neighbourhood. For the proof we replace in (17) π_M by any Lipschitz retraction.

Now we can prove our second main statement mentioned in the introduction.

Proof of Theorem B. Suppose, by contradiction, that P has no solutions. Using Lemma 3.11, take an open neighbourhood V of M in $\mathbb{R}^n$ with $dist(M, \mathbb{R}^n \setminus V) = \varepsilon > 0$ and such that h has an extension k on V which is still a Lipschitz selection of F. Let $v : \mathbb{R}^n \to [0, 1]$ be a Lipschitz function defined by $v(x) := \max\{0, 1 - \frac{1}{\varepsilon}\, dist(x, \mathbb{R}^n \setminus V)\}$. Then $v(x) = 0$ on M and $v(x) = 1$ on $\mathbb{R}^n \setminus V$.

Define the following Lipschitz map $G : \mathbb{R}^n \multimap \mathbb{R}^n$,

$$G(x) := (1 - v(x))k(x) + v(x)F(x)$$

which has the following properties:

(i) $G(x) = h(x)$ on M,

(ii) each trajectory for G leaves K through K_s, and $K_s(G) = K_s(F)$,

(iii) $G(x) \subset F(x)$, for every $x \in \mathbb{R}^n$, and hence, there is no solution to problem P for G.

Now, Proposition 3.1 used for G ends the proof. ∎

Remark 3.13. According to Remark 3.12, the above theorem is true under weaker assumption that M is a Lipschitz retract.

The aim of our further considerations is to check where we are able to find a Lipschitz selection of $F(\cdot) \cap T_K(\cdot)$ on $K_e \setminus K_s$. We describe some preliminary observations as lemmas.

LEMMA 3.14 (cf. Valadier [29]). *Let X be a (metric) space and $F, G : X \multimap \mathbb{R}^n$ two l.s.c. maps with closed convex values. Assume that $M \subset X$ is compact and $Int(F(x) \cap G(x)) \neq \emptyset$, for every $x \in M$.*

Then there is a Lipschitz selection f of $F(\cdot) \cap G(\cdot)$ on some open neighbourhood of M in X.

Proof. For any $x \in M$ choose $y_x \in \Psi(x) := F(x) \cap G(x)$ such that $B(y_x, \delta(x)) \subset \Psi(x)$, for some $\delta(x) > 0$. We show that there is an open neighbourhood U_x of x in X such that $y_x \in \Psi(z)$, for every $z \in U_x$. We shall do this for F (instead of Ψ), since the proof for G is the same.

Take a neighbourhood $V_x \ni x$ such that $F(x) \subset F(z) + B(0, \delta(x)/2)$, for every $z \in V_x$ (F is l.s.c.) and suppose, on the contrary, that for any $n \geqslant 1$ there is $z_n \in X$ such that $d(x, z_n) < \frac{1}{n}$ and $y_x \notin F(z_n)$.

Then, for n sufficiently large, $z_n \in V_x$ and $y_x \notin F(z_n)$. From the Separation Theorem for convex sets, it follows that there is a hyperplane separating y_x and $F(z_n)$, which implies that we can find $|y - y_x| < \delta(x)$ and an element y with $d_{F(z_n)}(y) > \delta(x)/2$. But $y \in F(x)$; a contradiction.

We have the covering $\{U_x\}_{x \in M}$ of M in X and we can choose a finite subcovering $\mathscr{U} = \{U_1, \ldots, U_k\}$, since M is compact. Take a Lipschitz partition of unity $\{a_i(\cdot)\}$ subordinated to $\mathscr{U}$, and define $f : \bigcup_{i=1}^k U_i \to \mathbb{R}^n$,

$$f(x) := \sum_{i=1}^{k} a_i(x) y_{x_i}.$$

It is easy to see that f is Lipschitz. Moreover, since values of Ψ are convex, f is a selection of Ψ. ∎

Note that for two Lipschitz maps with compact convex values, the intersection map may not be Lipschitz and may not even be l.s.c. Nevertheless, the following fact is true.

LEMMA 3.15. *Let M be a subset of a $C^{1,1}$ $(n-1)$-manifold X in $\mathbb{R}^n$ and $F : M \multimap \mathbb{R}^n$ a Marchaud locally Lipschitz map satisfying*:

(F1) *for each $x \in M$ and $y \in \partial F(x)$, the cone $T_{F(x)}(y)$ is a half-space*;
(F2) *for the hamiltonian $\mathscr{H}(x,p) := \min\{\langle v, p\rangle \mid v \in F(x)\}$, the derivative $\frac{\partial \mathscr{H}}{\partial p}(x,p)$ exists*[7] *and is locally Lipschitz on $M \times (\mathbb{R}^n \setminus \{0\})$*;
(F3) *$F(x) \cap T_X(x) \neq \emptyset$ for every $x \in M$.*

Then there exists a locally Lipschitz selection f of $\Psi(\cdot) := F(\cdot) \cap T_X(\cdot)$.

For the proof see Appendix B.

We collect lemmas above and apply to prove the following.

THEOREM 3.16. *Let K be a closed subset of $\mathbb{R}^n$ and $F : \mathbb{R}^n \multimap \mathbb{R}^n$ be a Marchaud locally Lipschitz map such that K_s is closed, $M = \overline{K_e \setminus K_s}$ is a compact proximal retract and the following conditions are satisfied*:

(i) *the map $T_K(\cdot)$ is l.s.c. on M*;

(ii) *there is an open neighbourhood X of $N := K_s \cap M$ in ∂K which is a $C^{1,1}$ $(n-1)$-manifold, and* (F1)–(F2) *holds for X and $X \cap M$*;

(iii) *$Int(F(x) \cap T_K(x)) \neq \emptyset$, for every $x \in M \setminus N$*;

If K_s is not a strong deformation retract of K, then problem P *has a solution.*

[7] Remark that $\frac{\partial \mathscr{H}}{\partial p}(x,p)$ is the unique point in $F(x)$ such that $\frac{\partial \mathscr{H}}{\partial p}(x,p) = \operatorname{Arg\,Min}_{v \in F(x)} \langle v, p \rangle$.

Proof. At first, assumption (ii) implies that on some open neighbourhood W_1 of N in M the map $T_{\partial K}(\cdot)$, and hence $\Psi(\cdot) := F(\cdot) \cap T_{\partial K}(\cdot)$ are u.s.c. Thus $\Psi(x) \neq \emptyset$ for every $x \in N$.

Take an open set W in M with $N \subset W \subset \bar{W} \subset X \cap W_1$. From Lemma 3.15 it follows that there is a Lipschitz selection g_1 of Ψ on $\bar{W}$. Let V be an open neighbourhood of N in M with $N \subset \bar{V} \subset W$. By Lemma 3.14, there is a Lipschitz selection g_2 of $\Psi_1 := F(\cdot) \cap T_K(\cdot)$ on $M \setminus V$.

Since N and $M \setminus V$ are compact, $dist(N, M \setminus V) = \varepsilon > 0$. Take $\Omega := \{x \in M \mid dist(x, N) \leqslant \frac{\varepsilon}{2}\}$ and define an Urysohn (Lipschitz) function $v : M \to [0, 1]$,

$$v(x) := \max\left\{0, 1 - \frac{2}{\varepsilon} dist(x, M \setminus W)\right\}.$$

Notice that $v(x) = 1$ on N and $v(x) = 0$ on $M \setminus V$.

Consider the map $g : M \to \mathbb{R}^n$,

$$g(x) := v(x) g_1(x) + (1 - v(x)) g_2(x).$$

One can see that g is a Lipschitz selection of $F(\cdot) \cap T_K(\cdot)$ on M.

Using Theorem B we end the proof. ∎

COROLLARY 3.17. *Let K be a closed subset of $\mathbb{R}^n$ with ∂K being a $C^{1,1}$ $(n-1)$-manifold and F be a Marchaud locally Lipschitz map which satisfies* (F1)–(F2) *on $M := K_e \setminus K_s$. Assume that K_s is closed and M is a compact proximal retract.*

If K_s is not a strong deformation retract of K, then problem P *has a solution.*

Proof. It is obvious that $\Psi(x) := F(x) \cap T_{\partial K}(x) \neq \emptyset$ for every $x \in K_e \setminus K_s$. The same argument as in the previous proof implies also that $\Psi(x) \neq \emptyset$ on $K_s \cap M$. As an open neighbourhood X of $K_s \cap M$ in ∂K satisfying assumption (ii) in Theorem 3.16 we take ∂K. Then, without using assumption (iii), we can find, by Lemma 3.15, a Lipschitz selection g of Ψ on M. The statement follows from Theorem B. ∎

Remark 3.18. Consider the class of maps $F : \mathbb{R}^n \multimap \mathbb{R}^n$ of the form

$$F(x) = f(x) + A(x)B,$$

where B is a closed ball in $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a single-valued locally Lipschitz map and $A \in L_{\text{lip}}(\mathbb{R}^n, \mathbb{R}^n)$, which means that A is a locally Lipschitz map defined on $\mathbb{R}^n$ with values being linear automorphisms of $\mathbb{R}^n$.

One can check that each map F of the above form satisfies (F1) and (F2). This leads us to the following application in control theory.

COROLLARY 3.19. *Consider the following control problem*:

$$\begin{cases} \dot{x} = f(x) + A(x)u, \\ u \in U = cl\, B(0,1), \end{cases} \tag{18}$$

where K is a closed set with ∂K being a $C^{1,1}$ $(n-1)$-manifold $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz and $A \in L_{\text{lip}}(\mathbb{R}^n, \mathbb{R}^n)$.

If $M := K_e \backslash K_s$ is a compact proximal retract, K_s is closed, but not a strong deformation retract of K, then there exists at least one control u such that problem P *has a solution to* (18).

4. ON THE SUBSET OF INITIAL CONDITIONS $x_0 \in K$ SOLVING P

Let K be a closed subset of $\mathbb{R}^n$ and $x_0 \in K$ be an arbitrary point. We say that $x \in S_F(x_0)$ is *viable in* K provided $x(t) \in K$, for all $t \geqslant 0$. A set K is said to be *viable under* F, if for any initial point $x_0 \in K$ there exists at least one viable solution in K starting from x_0.

DEFINITION 4.1. The *viability kernel of K for F* (denoted by $Viab_F(K)$) is the set of all initial points $x_0 \in K$ such that problem P has a solution (at least one viable solution starts from x_0). It is also (see [3, Theorem 4.1.2]), the largest closed subset of K viable under F.

Of course, in general a viability kernel may be empty. All previous results of the present paper can be viewed as sufficient conditions for nonemptiness of $Viab_F(K)$.

The following result gives an information on a topological structure of a viability kernel.

PROPOSITION 4.2. *Let K be a closed subset of $\mathbb{R}^n$ and $F : \mathbb{R}^n \multimap \mathbb{R}^n$ a Lipschitz Marchaud map such that $K_s = K_e$ is closed. Then $\check{H}(K \backslash Viab_F(K))$ and $\check{H}(K_e)$ are isomorphic.*[8]

Proof. Define the family of sets

$$A_m := cl\{x_0 \in K \mid \forall x \in S_F(x_0)\colon\ \tau_K(x) \leqslant m\}, \quad m \geqslant 1.$$

We show that $A_m \cap Viab_F(K) = \emptyset$, for every $m \geqslant 1$.

Suppose, on the contrary, that there exists $x_0 \in A_m \cap Viab_F(K)$. It means that there is a viable solution $v \in S_F(x_0)$ and a sequence $\{x_k\} \subset K$ converging to x_0 with $\tau_K(z) \leqslant m$, for each $z \in S_F(x_k)$ and $k \geqslant 1$.

[8] We do not assume a priori that the viability kernel is nonempty.

Since F is Lipschitz, we can find a sequence of $z_k \in S_F(x_k)$ converging to $v \in S_F(x_0)$. Without any loss of generality, we can assume that $\tau_K(z_k) \to t \leqslant m$. Then

$$|v(t) - z_k(\tau_K(z_k))| \leqslant |v(t) - v(\tau_K(z_k))| + |v(\tau_K(z_k)) - z_k(\tau_K(z_k))| \to 0,$$

which implies that $v(t) \in K_s$; a contradiction.

Since for each $x_0 \in K \setminus Viab_F(K)$ we can find $m \geqslant 1$ such that each trajectory starting from x_0 leaves K before m, we deduce that $K \setminus Viab_F(K) = \bigcup_{m=1}^{\infty} A_m$. Moreover, $A_m \subset A_{m+1}$, for every $m \geqslant 1$.

Notice that the sets A_m form a direct system (with respect to the semi-order induced by the inclusion), and for every compact subset $B \subset K \setminus Viab_F(K)$ there is some A_m with $B \subset A_m$. Hence, one obtains (see e.g. [11, pp. 278–280]) that

$$\check{H}(K \setminus Viab_F(K)) \cong \lim_{\rightarrow} \{\check{H}(A_m)\}, \tag{19}$$

where $\cong$ denotes an isomorphism and $\lim_{\rightarrow}$ a direct limit.

To compute $\check{H}(A_m)$, notice that each trajectory starting from A_m leaves A_m through K_e and we can follow the proof of Theorem A or Proposition 3.1 for A_m to obtain that $\check{H}(A_m) \cong \check{H}(K_e)$. Combining with (19) we conclude that $\check{H}(K \setminus Viab_F(K)) \cong \check{H}(K_e)$, which ends the proof. ∎

Remark 4.3. The statement of the above proposition implies that if $\check{H}(K)$ and $\check{H}(K_e)$ are not isomorphic, then there is a viable solution in K. Indeed, otherwise $\check{H}(K) = \check{H}(K \setminus Viab_F(K)) \cong \check{H}(K_e)$; a contradiction.

COROLLARY 4.4 (Cardaliaguet [9, Theorem 2.1]). *Under assumptions of Proposition* 4.2, *each connected component of* $K \setminus Viab_F(K)$ *contains one and only one component of* K_e.

Note that this result was directly proved in [8] for arbitrary Marchaud maps when K_s is closed and assumption (10) is satisfied.

Below we shall show that in some special cases one can say more about a topological structure of a viability kernel. We will need the following result.

LEMMA 4.5 (Hyman [18]). *Let M be a compact subset of an absolute neighbourhood retract Y. Then M is R_δ if and only if M is contractible in every open neighbourhood in Y.*

It is well known that every retract of $\mathbb{R}^n$ is an absolute neighbourhood retract.

PROPOSITION 4.6. *Let $K = \overline{Int\, K} \subset \mathbb{R}^n$ be compact. Assume that K is a retract of $\mathbb{R}^n$ and F is a Lipschitz Marchaud map satisfying*

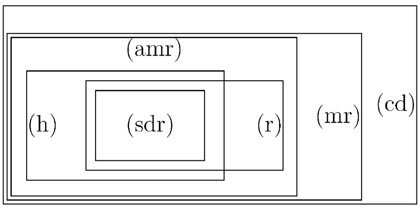
$$F(x) \cap T_K(x) = \emptyset, \quad \textit{for every } x \in \partial K.$$

Then $Viab_F(K)$ is a compact R_δ-set.

Proof. From Corollary 3.3 it follows that $Viab_F(K) \neq \emptyset$. It is obviously contained in *Int* K. By Proposition 3.2 in [9], for every open neighbourhood V of $Viab_F(K)$ there exists a compact set $M \subset V$ such that $Viab_F(K) \subset M$ and M is a retract of K. Since K is a retract of $\mathbb{R}^n$, each set M is contractible. Applying Lemma 4.5 we end the proof. ∎

5. CONCLUDING REMARKS

For clarifying the difference between main parts of the present paper, we describe relations between topological conditions arising in main results of the paper in the following picture:



where simplifying that K is compact and $K_s = K_e$,

(cd) K is connected and K_e is disconnected,
(mr) K_e is a multivalued retract of K,
(amr) K_e is an admissible multivalued retract of K,
(r) K_e is a retract of K,
(sdr) K_e is a strong deformation retract of K,
(h) the inclusion $i : K_e \hookrightarrow K$ induces the isomorphism $i_* : \check{H}(K_e) \to \check{H}(K)$.

Finally note that, according to Lemma 1.3, we can analogously as for F, consider K_e for $-F$ (for a compact set K) and study its topological relation with K. Sometimes it may occur that this new situation is easier for applying the results presented in the paper.

APPENDIX A. ADMISSIBLE MAPS

Below we give some notions and properties concerning so-called admissible multivalued maps in the sense of Górniewicz [15]. For more details see [16].

A map $p : X \to Y$ is said to be a *Vietoris map* provided p is onto, proper (i.e. $p^{-1}(A)$ is compact, for any compact subset A of Y), and the set $p^{-1}(y)$ is acyclic[9] for any $y \in Y$.

DEFINITION A.1. A multivalued map $\varphi : X \multimap Y$ is called *admissible* provided there exists a space Γ and two single-valued maps $p : \Gamma \to X$ and $q : \Gamma \to Y$ such that

(i) p is a Vietoris map,
(ii) $q(p^{-1}(x)) \subset \varphi(x)$, for every $x \in X$.

We will say that the pair (p, q) above is a *selected pair* of φ and denote it by $(p, q) \subset \varphi$. Of course, φ may have many selected pairs. From the Vietoris theorem (see [16, Theorem 8.9]) it follows that a Vietoris map p induces an isomorphism $p_* : \check{H}(X) \to \check{H}(Y)$.

It allows us to consider for any selected pair (p, q) of φ a homomorphism

$$\check{H}(X) \xrightarrow{p_*^{-1}} \check{H}(\Gamma) \xrightarrow{q_*} \check{H}(Y).$$

We define

$$\varphi_* := \{q_* p_*^{-1} \mid (p, q) \subset \varphi\}.$$

Remark A.2. Note that every *acyclic* map, i.e. a multivalued map $\varphi : X \multimap Y$ with compact acyclic values, is admissible. In fact, we have a selected pair $X \xleftarrow{p_\varphi} Gr(\varphi) \xrightarrow{q_\varphi} Y$, $p_\varphi(x, y) = x, q_\varphi(x, y) = y$.

If $\varphi : X \multimap Y$ is acyclic, then for any two selected pairs $(p, q), (p', q')$ of φ, $q_* p_*^{-1} = q'_*(p')_*^{-1}$ [16, Proposition 40.4]. Hence, since every single-valued map is acyclic, we can obtain f_*, for $f : X \to Y$, considering both diagrams $X \xleftarrow{p_f} Gr(f) \xrightarrow{q_f} Y$ and $X \xleftarrow{id_X} X \xrightarrow{f} Y$.

[9] With respect to the Čech homology functor with compact carriers and coefficients $\mathbb{Q}$ (see, e.g., [12]). It means that $\check{H}_n(p^{-1}(y)) = 0$ for $n > 0$ and $\check{H}_0(p^{-1}(y)) = \mathbb{Q}$.

PROPOSITION A.3 (Górniewicz [16, Theorem 40.5]). *Let $\varphi : X \multimap X_1$ and $\psi : X_1 \multimap X_2$ be two admissible maps. Then the composition $\psi \circ \varphi : X \multimap X_2$ is an admissible map and, for each selected pairs $(p_1, q_1) \subset \varphi$ and $(p_2, q_2) \subset \psi$, there exists a selected pair $(p, q) \subset \psi \circ \varphi$ such that $(q_2)_*(p_2)_*^{-1}(q_1)_*(p_1)_*^{-1} = q_* p_*^{-1}$.*

We will need the following.

PROPOSITION A.4. *Let $\varphi : X \multimap X_1$ and $\psi : X_1 \multimap X_2$ be two admissible maps. If $\psi = i : X_1 \hookrightarrow X_2$ (resp. $\varphi = i : X \hookrightarrow X_1$), then*

$$(i \circ \varphi)_* = \{ i_* q_* p_*^{-1} \mid (p, q) \subset \varphi \}, \tag{A.1}$$

resp.

$$(\psi \circ i)_* = \{ q_* p_*^{-1} i_* \mid (p, q) \subset \psi \}. \tag{A.2}$$

Proof. By Theorem 40.5 in [16], if $\varphi = i : X \hookrightarrow X_1$ (resp. $\psi = i : X_1 \hookrightarrow X_2$), then, for any selected pair $(p_2, q_2) \subset \psi$ (resp. $(p_1, q_1) \subset \varphi$), there is a selected pair $(p, q) \subset \psi \circ i$ (resp. $(p, q) \subset i \circ \varphi$) such that

$$(q_2)_*(p_2)_*^{-1} i_* = q_* p_*^{-1}, \tag{A.3}$$

resp.

$$i_*(q_1)_*(p_1)_*^{-1} = q_* p_*^{-1}. \tag{A.4}$$

We show that also inversely, if moreover $\psi : X_1 \multimap X$ (resp. $\varphi : X_2 \multimap X_1$), then for every selected pair $(p, q) \subset \psi \circ i$ (resp. $(p, q) \subset i \circ \varphi$) there is a selected pair $(p_2, q_2) \subset \psi$ (resp. $(p_1, q_1) \subset \varphi$) such that (A.3) and (A.4) hold.

Let $X \overset{p}{\leftarrow} \Gamma \overset{q}{\rightarrow} X$ and $X_1 \overset{\bar{p}}{\leftarrow} \bar{\Gamma} \overset{\bar{q}}{\rightarrow} X$ be arbitrary selected pairs of $\psi \circ i$ and ψ, respectively. Consider a free topological union $\Gamma_0 = \Gamma + \bar{p}^{-1}(X_1 \backslash X)$ and the following maps $p_2 : \Gamma_0 \to X_1$ and $q_2 : \Gamma_0 \to X$,

$$p_2(z) := \begin{cases} p(z) & \text{for } z \in \Gamma, \\ \bar{p}(z) & \text{for } z \in \bar{p}^{-1}(X_1 \backslash X), \end{cases}$$

$$q_2(z) := \begin{cases} q(z) & \text{for } z \in \Gamma, \\ \bar{q}(z) & \text{for } z \in \bar{p}^{-1}(X_1 \backslash X). \end{cases}$$

One can check that p_2 is a Vietoris map and q_2 is continuous. Therefore we have the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "id_X"'] & X_1 & \Gamma_0 \arrow[l, "p_2"'] \arrow[d, "q_2"] \\
X & \Gamma \arrow[l, "p"'] \arrow[r, "q"] \arrow[ul, "p"] \arrow[u, "i \circ p"'] \arrow[ur, "f"'] & X
\end{tikzcd}$$

where $f(z) = z \in \Gamma_0$, for $z \in \Gamma$.

Applying for this diagram the functor $\check{H}$, we obtain (A.3) and hence (A.2).

Now, for any selected pair $X_2 \overset{p}{\leftarrow} \Gamma \overset{q}{\rightarrow} X_2$ of $i \circ \varphi$, notice that all values of q lie in X_1, so we can consider the map $q_1 : \Gamma \to X_1$, $q_1(z) = q(z)$. Denote also $p_1 := p$. Then $(p_1, q_1) \subset \varphi$ and $q(p^{-1})(x) = i(q_1(p_1^{-1}))(x)$, for every $x \in X_2$.

Commutativity of the diagram

$$\begin{tikzcd}
\Gamma \arrow[r, "q_1"] \arrow[d, "p_1"'] & X_1 & X_1 \arrow[l, "id_{X_1}"'] \arrow[d, "i"] \\
X_2 & \Gamma \arrow[l, "p"'] \arrow[r, "q"] \arrow[ul, "id_\Gamma"] \arrow[u, "q_1"'] \arrow[ur, "q_1"'] & X_2
\end{tikzcd}$$

implies (A.4) and thus (A.1). ∎

Since a composition of two admissible maps is admissible, the class of admissible maps is large and contains many operators arising in differential problems. In particular, the Poincaré operator for a differential inclusion $ev_T \circ S_F(\cdot)$, where $ev_T(x) := x(T)$, with a Marchaud right-hand side F is admissible (also in functional differential problems).

It is easy to see that, for any two admissible maps, if $\varphi \subset \psi$, then $\varphi_* \subset \psi_*$.

Two admissible maps $\varphi, \psi : X \multimap Y$ are *homotopic* (written $\varphi \sim \psi$) provided there exists an admissible map $\chi : X \times [0,1] \multimap Y$ such that $\chi(\cdot, 0) \subset \varphi$ and $\chi(\cdot, 1) \subset \psi$.

PROPOSITION A.5 (Górniewicz [16, Theorem 40.11, Corollary 40.12]). *For any two admissible maps* $\varphi, \psi : X \multimap Y$, *if* $\varphi \sim \psi$, *then* $\varphi_* \cap \psi_* \neq \emptyset$.

In particular, if $\varphi : X \multimap X$ and $\varphi \sim id_X$, then $Id_{\check{H}(X)} = q_* p_*^{-1}$, for some selected pair (p, q) of φ.

APPENDIX B. PROOF OF LEMMA 3.15

Let $x \in M$ be an arbitrary point. Since X is a $C^{1,1}$ $(n-1)$-manifold, there is an open neighbourhood U_x of x in $\mathbb{R}^n$ and a map $\phi : U_x \to \mathbb{R}^n$ such that $\phi|_{U_x \cap X} : U_x \cap X \to V_x \subset \mathbb{R}^{n-1}$ is a homeomorphism, for V_x open in $\mathbb{R}^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_n = 0\}$, $\phi'(\cdot)$, $(\phi'(\cdot))^{-1}$ exist and are Lipschitz on $U_x \cap X$, and moreover, the following conditions hold:

$$w \in T_X(z) \text{ if and only if } \phi'(z)(w) \in T_{V_z}(\phi(z)) = \mathbb{R}^{n-1}, \tag{B.1}$$

$$w \in (T_X(z))^- \text{ if and only if } \phi'(z)(w) \in (\mathbb{R}^{n-1})^-. \tag{B.2}$$

Define $m(z) := \phi'(z)^{-1}(0, \ldots, 0, 1)$. By (B.2), $m(z) \in (T_X(z))^-$. Define also

$$v^+(z) := \operatorname*{Arg\,Min}_{v \in F(z)} \langle v, m(z) \rangle = \frac{\partial \mathscr{H}}{\partial p}(z, m(z))$$

and

$$v^-(z) := \operatorname*{Arg\,Max}_{v \in F(z)} \langle v, m(z) \rangle = -\frac{\partial \mathscr{H}}{\partial p}(z, -m(z)).$$

Lipschitzianity of $m(\cdot)$ and assumptions (F1) and (F2) imply that also $v^+(\cdot)$ and $v^-(\cdot)$ are Lipschitz. Moreover, $Int\, F(z) \neq \emptyset$ and therefore, $[v^-(z), v^+(z)] \not\subset T_X(z)$ for every $z \in U_x \cap X$. By the convexity of $F(z)$, it follows that $[v^-(z), v^+(z)] \subset F(z)$.

Define

$$f(z) := [v^-(z), v^+(z)] \cap T_X(z).$$

Obviously, f is a selection of Ψ.

We shall show that f is Lipschitz on some open neighbourhood of x in X. To this end, notice that, by (B.1)

$$\begin{aligned} f(z) &\in [v^-(z), v^+(z)] \cap T_X(z) \\ &\Leftrightarrow \phi'(z)(f(z)) \in [\phi'(z)(v^-(z)), \phi'(z)(v^+(z))] \cap \mathbb{R}^{n-1}. \end{aligned}$$

Denote $u^-(z) := \phi'(z)(v^-(z))$, $u^+(z) := \phi'(z)(v^+(z))$ and $b(z) := \phi'(z)(f(z))$. Then $f(z) = \phi'(z)^{-1}(b(z))$ and, since $(\phi'(\cdot))^{-1}$ is also Lipschitz, it is sufficient to check that $b(\cdot)$ is Lipschitz.

Notice that

$$b(z) = u^-(z) + \lambda(z)(u^+(z) - u^-(z)) \qquad \text{for any } z \in U_x \cap X, \tag{B.3}$$

where $\lambda(z) \in [0, 1]$.

Since $u^-(x) \neq u^+(x)$ and $[u^-(x), u^+(x)] \not\subset \mathbb{R}^{n-1}$, it follows that there is an open neighbourhood $V_x \subset U_x$ of x such that

$$|u_n^+(z) - u_n^-(z)| \geqslant \delta > 0 \qquad \text{for every } z \in V_x \cap X,$$

where $u_n^+(z)$ and $u_n^-(z)$ denote nth components of $u^+(z)$ and $u^-(z)$, respectively.

But nth component $b_n(z)$ of $b(z)$ is equal to 0 and hence,

$$\lambda(z) = \frac{|u_n^-(z)|}{|u_n^+(z) - u_n^-(z)|},$$

which implies that $\lambda(\cdot)$, and so $b(\cdot)$, are Lipschitz on V_x. The proof is complete. ∎

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